THE α-INVARIANTS ON TORIC FANO MANIFOLDS

Jian Song

Department of Mathematics
Columbia University, New York, NY 10027

§1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian[14], Tian and Yau[13] is closely related to the existence of Kähler-Einstein metrics. In his solution to the Calabi conjecture, Yau[19] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with nonpositive first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there exist known obstructions such as the Futaki invariant. For a compact Kähler manifold M with positive first Chern class, Tian[14] proved that M admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$[16].

There have been many nice results on the classification of toric Fano manifolds. Mabuchi discovered that if a toric Fano manifold is Kähler-Einstein then the barycenter of the polyhedron defined by its anticanonical divisor is at the origin. V. Batyrev and E. Selivanova [2] estimate the lower bound of α-invariant of symmetric toric Fano manifolds which is sufficient to show the existence of Kähler-Einstein metric.

In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman kernel on polarized Kähler metrics to approximate plurisubharmonic functions and obtain a formula to calculate the α_G-invariants of all toric Fano manifolds precisely. This gives a generalization of the result by V. Batyrev and E. Selivanova[2] and also this formula confirms the earlier result [12] on the estimates of α invariants on $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$.

Our main theorems are

Theorem 1.1 *If X is a toric Fano manifold of complex dimension n then*
(a) $\alpha_G(X) = 1$ *if X is symmetric, otherwise*
(b) $\alpha_G(X) = \frac{\min_{0\neq v\in S}\frac{\lfloor w_v \rfloor}{|v|}}{1+\min_{0\neq v\in S}\frac{\lfloor w_v \rfloor}{|v|}} \leq \frac{1}{2}$.

Corollary 1.1 *If X is a toric Fano manifold, then X is symmetric if and only if $\alpha_G(X) = 1$.*

Theorem 1.2 *If X is a toric Fano manifold, then $\{\alpha_{m,G}(X)\}_{m\geq 1}$ is stationary. More precisely, $\alpha_{m,G}(X) = \alpha_G(X)$ if $m \geq m_0$, where m_0 is the least positive integer such that $m_0 v$ is an integral point and v is the minimizer of* $\min_{0\neq v\in S} \frac{|w_v|}{|v|}$.

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§2. Notations

Let us first recall the definition of an invariant $\alpha_G(X)$ introduced by Tian. Let X be an n-dimensional compact complex manifold with positive first Chern class $c_1(X)$ and G a compact subgroup of $Aut(X)$. Choose a G-invariant Kähler metric $g = g_{i\bar{j}}$ on X such that $\omega_g = \frac{\sqrt{-1}}{2\pi}\sum g_{i\bar{j}} dz_i \wedge d\overline{z}_j \in c_1(X)$.

Definition Let $P_G(X, g)$ be the set of all C^2-smooth G-invariant real-valued functions φ such that $\sup_X \varphi = 0$ and $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0$. The $\alpha_G(X)$ invariant is defined as superemum of all $\lambda > 0$ such that

$$\int_X e^{-\alpha\varphi}\omega^n \leq C(\alpha)$$

for all $\varphi \in P_G(X, g)$, where $C(\alpha)$ is a positive constant depending only on α, g and X.

Let N be a lattice of rank n, $M = Hom(N, Z)$ the dual lattice. $M_R = M \otimes_Z R$, $N_R = N \otimes_Z R$. Let $X = X_\Sigma$ be a smooth projective toric $n-$fold defined by a complete fan Δ of regular cones $\Delta \subset M_R$ and denote $\Delta(i)$ the i-dimensional cone of Δ. We put $T = C^* =\{(t_1, t_2, ..., t_n)|t_i \in C^*\}$. For $a \in M$ and $b \in N$, we define $< a, b >\in Z$, $\chi^a \in Hom_{alggp}(T, C^*)$ by

$$< a, b >= \sum_{i=1}^{n} a_i b_i,$$
$$\chi^a((t_1, ..., t_n)) = t_1^{a_1} t_2^{a_2} ... t_n^{a_n}.$$

For each $\rho \in \Delta(1)$, let b_ρ denote the uniquie fundamental generator of ρ. We now consider the divisor $K = -\sum_{\rho\in\Delta(1)} D(\rho)$ on $X = X_\Delta$. The following theorem is due to Demazure[4].

Theorem 2.3 *K is a canonical divisor of X_Δ, and the following are equivalent:*
(a) X_Δ is a toric fano manifold.
(b) $-K$ is ample.
(c) $-K$ is very ample.
(d) $\Sigma_{-K} =\{\ a \in M_R|< a, b_\rho >\leq 1$ for all $\rho \in \Delta(1)\}$ is an n-dimensional compact convex polyhedron whose vertices are exactly $\{a_\tau|\tau \in \Delta(n)\}$, where each a_τ denotes the unique element of M such that $< a_\tau, b >= 1$ for all fundamental generators b of τ.

The maximal torus $T \subset Aut(X)$ acting on X has an open dense orbit $U \subset X$, so the normalizer $N(T) \subset Aut(X)$ of T has a natural action on U. Let $W(X) = N(T)/T$ and we identify the maximal torus $T \subset Aut(X)$ with an open dense orbit U in X by choosing an arbitrary point$x_0 \in U$, then we have the following splitting short exact sequence

$$1 \to T \to N(T) \to W(X) \to 1,$$

i.e., an embedding $W(X) \hookrightarrow N(T)$. Denote by $K(T) = (S^1)^n$ the maximal compact subgroup in T. We choose G to be the maximal compact subgroup in $N(T)$ generated by $W(X)$ and $K(T)$, so that we have the short exact sequence

$$1 \to K(T) \to G \to W(X) \to 1.$$

Proposition 2.1 *Let X=X_Δ be a smooth projective toric n-fold defined by a complete regular polyhedral fan Δ. Then the group W(X) is isomorphic to the finite group of all symmetries of Δ, i.e., W(X) is isomorphic to a subgroup of $GL(N)(\simeq GL(n, Z))$ consisting of all elements $\gamma \subset GL(N)$ such that $\gamma(\Delta) = \Delta$.*

Remark: $W(X)$ is as well isomorphic to a subgroup of $GL(M)(\simeq GL(n, Z))$ consisting of all elements $\gamma \subset GL(M)$ such that $\gamma(\Sigma) = \Sigma$.

Definition A toric n-fold X is symmetric if the trivial character is the only $W(X)$-invariant algebraic character of T, i.e. $N^{W(X)} = \{\chi \in N|\ g\chi = \chi$ for all $g \in W(X)\}$=$\{0\}$.

Definition Let $S = \{v \in \partial\Sigma|\ gv = v$ for all $g \in W(X)\}$ be the stable points of $W(X)$ on the boundary of Σ. If $S \neq \{0\}$ then for any $0 \neq v \in S$, we define w_v related with v by $w_v = \partial\Sigma \cap \{-tv|\ t \geq 0\}$.

Remark: It's easy to see X is symmetric if and only if $S=0$.

§3. Holomorphic approximation of PSH

In this section, we will employ the technique in [15, 20] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem[20].

Theorem 3.4 *Let M be a compact complex manifold of dimension n and let $(L,h)\to M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m\in N$, h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, S_1^m, ..., S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M,L^m)$, $d_m=\dim H^0(M,L^m)$, with respect to the inner product:*

$$(S_1,S_2)_{h_m} = \int_M h_m(S_1(x),S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + ...$$

for some smooth coefficients $a_j(x)$ with $a_0=1$. More precisely, for any k:

$$\left\|\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 - \sum_{j<R} a_j(x)m^{n-j}\right\|_{C^k} \leq C_{R,k}m^{n-R}$$

where $C_{R,k}$ depends on R,k and the manifold M.

Let

$$\begin{aligned} \tilde{\omega}_g &= \omega_g + \sqrt{-1}\partial\overline{\partial}\phi > 0 \\ \widetilde{h} &= he^{-\phi} \end{aligned}$$

Let $\widetilde{h}_m$ be the induced Hermitian metric of $\widetilde{h}$ on L^m, $\{\widetilde{S}_0^m, \widetilde{S}_{1,...,}^m \widetilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M,L^m)$, where $d_m=\dim H^0(M,L^m)$, with respect to the inner product

$$(S_1,S_2)_{\widetilde{h}_m} = \int_M \widetilde{h}_m(S_1(x),S_2(x))dV_{\widetilde{g}}\,.$$

By Theorem 3.4, we have

$$\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m} = \left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right) e^{-m\phi}.$$

Thus

$$\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right) = -\frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right)$$

As $m \to +\infty$, we obtain for any positive integer R

$$\begin{aligned} & \frac{1}{m}\log\left(\sum_{j<R} \widetilde{a}_j(x) m^{n-j}\right) \\ = & \frac{1}{m}\log m^n (\sum_{j<R} \widetilde{a}_j(x) m^{-j}) \\ = & \frac{n}{m}\log m + \frac{1}{m}\log(1+O(\frac{1}{m})) \to 0 \end{aligned}$$

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

Corollary 3.2

$$\left\| \phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||^2_{h_m}\right)\right\|_{C^k} \to 0, \ as\ m \to +\infty.$$

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of L^m.

§4. Proof of The Main Theorem

Suppose X_Δ is Fano, then one obtains a convex $W(X)$-invariant polyhedron Σ in M_R defined by $\Sigma = \{\ a \in M_R \mid < a, b_\rho > \leq 1$, for all $\rho \in \Delta(1)\ \}$ where b_ρ is the fundamental generator of ρ. Let $L(\Sigma) = \{v_0, v_1, ..., v_k\} = M \cap \Sigma$. Then $v_0, v_1, ..., v_k$ determine algebraic characters $\chi_i : T \to C^*$ of T(i=0, 1,..., k). Moreover, we have

$$|\chi_i(x)|^2 = e^{<v_i, y>}, i = 0, ..., k,$$

where y is the image of x under the canonical projection $\pi : T \to M_R$. Let us define $u : U \to R$ as follows:

$$u = \log(\sum_{i=0}^{k} |\chi_i(x)|^2), x \subset U \simeq T.$$

Since u is $K(T)$-invariant, u descends to a function $\tilde{u} : M_R \to R$ defined as

$$\tilde{u} = \log(\sum_{i=0}^{k} e^{<v_i,y>}), y \subset M_R.$$

Consider the G-invariant hermitian metric $g = g_{i\bar{j}}$ on X such that the restriction of the corresponding to g differential 2-form on U is defined by

$$\omega_g = \partial\overline{\partial}u.$$

The metric g is exactly the pull-back of Fubini-Study metric from P^m with respect to the anticanonical embedding $X \hookrightarrow P^m$ defined by the algebraic characters $\chi_0, \chi_1, ..., \chi_k$.

Let $\Sigma^{(m)}$=$\{\ a \in M_R \mid <a, b_\rho> \leq m$ and $L(\Sigma^{(m)}) = \{v_0, ..., v_{k_m}\} = M \cap \Sigma^{(m)}$, where $k_m + 1 = \dim H^0(X, O((-K)^m))$ and χ^μ: $T \to C^*$ defined by $|\chi^\mu(x)|^2 = e^{<\mu,y>}$. We have the following lemma (see [7] p66)

Lemma 4.1 $H^0(X, O((-K)^m)) = \oplus_{\mu \in L(\Sigma^{(m)})} C \cdot \chi^\mu$.

Proposition 4.2 $\{\chi^\mu\}_{\mu \in L(\Sigma^{(m)})}$ *is an orthogonal basis of* $H^0(X, O((-K)^(m))$ *with respect to the inner product* $<,>_{h^m}$, *where* $h^m = \frac{1}{(\sum_{i=0}^{k} |\chi_i(x)|^2)^m}$.

Proof

$$\begin{aligned} & \int_X <\chi^\mu, \chi^\nu>_{h^m} \omega^n \\ = & \int_T \frac{(z_1^{\mu_1}...z_n^{\mu_n})(\overline{z}_1^{\nu_1}...\overline{z}_n^{\nu_n})}{(\sum_{i=0}^{k} |z^{v_i}|^m)} \omega^n \\ = & \int_T \frac{|z_1|^{\mu_1+\nu_1}...|z_n|^{\mu_n+\nu_n} e^{i(\mu_1-\nu_1)\theta_1}...e^{i(\mu_n-\nu_n)\theta_n}}{(\sum_{i=0}^{k} |z^{v_i}|^m)} \omega^n. \end{aligned}$$

which is 0 if $\mu \neq \nu$.

For any $\varphi \in P_G(X, \omega)$, by Corollary 3.3

$$\varphi(x) = \lim_{m\to\infty} \frac{1}{m} \log \frac{\sum_{\mu \in L(\Sigma^{(m)})} a_\mu^{(m)} |\chi^\mu(x)|^2}{(\sum_{i=0}^{k} |\chi_i(x)|^2)^k}$$

Lemma 4.2 *There exists* $\epsilon > 0$ *such that for* $\varphi \in P_G(X, \omega)$ *and* $\tilde{m} > 0$ *there exist* $m > \tilde{m}$ *and* $\mu \in L(\Sigma^{(m)})$ *with* $(a_\mu^{(m)})^{\frac{1}{m}} > \epsilon$.

Proof Otherwise, for any $\epsilon > 0$ there exists φ_ϵ and $\tilde{m}$ such that for any $m > \tilde{m}$ and $\mu \in L(\Sigma^{(m)})$ we have $(a_\mu^{(m)})^{\frac{1}{m}} < \epsilon$. By choosing m large enough we have

$$\begin{aligned}
\varphi_\epsilon(x) &\leq \frac{1}{m}\log\frac{\sum_{\mu\in L(\Sigma^{(m)})}|\chi^\mu(x)|^2}{(\sum_{i=0}^k|\chi_i(x)|^2)^m} + \log\epsilon \\
&= \frac{1}{m}\log(\sum_{\mu\in L(\Sigma^{(m)})}\frac{|\chi^\mu(x)|^2}{(\sum_{i=0}^k|\chi_i(x)|^2)^m}) + \log\epsilon \\
&\leq \frac{1}{m}\log(\sum_{\mu\in L(\Sigma^{(m)})} 1) + \log\epsilon \\
&\leq Const + \log\epsilon.
\end{aligned}$$

Since ϵ can be chosen arbitrarily small, the above inequality implies that $\varphi_\epsilon \to -\infty$ uniformly as ϵ goes to 0, which contradicts the fact that $\sup_X \varphi = 0$.

For any $\varphi \in P_G(X,\omega)$, by Lemma 4.1 we have

$$\begin{aligned}
\varphi(x) &= \lim_{m\to\infty}\frac{1}{m}\log\frac{\sum_{\mu\in L(\Sigma^{(m)})}a_\mu^{(m)}|\chi^\mu(x)|^2}{(\sum_{i=0}^k|\chi_i(x)|^2)^m} \\
&\geq \frac{1}{m}\log\frac{\sum_{g\in W(X)}|\chi^{g\mu}(x)|^2}{(\sum_{i=0}^k|\chi_i(x)|^2)^m} - C_1 \\
&\geq \log\frac{|\chi^{\sum_{g\in W(X)}g\mu}(x)|^{\frac{2}{m|W(X)|}}}{(\sum_{i=0}^k|\chi_i(x)|^2)} - C_1 \\
&\geq \log\frac{|\chi(x)|^{\frac{2\sum_{g\in W(X)}g\mu}{m|W(X)|}}}{(\sum_{i=0}^k|\chi_i(x)|^2)} - C_1
\end{aligned}$$

Put $v = \frac{\sum_{g\in W(X)}g\mu}{|W(X)|K}$, then we have $\tilde{\varphi}(y) \geq \log\frac{e^{<v,y>}}{\sum_{i=0}^m e^{<v_i,y>}}$
Put $y_i = \log|t_i|^2$ $t_i = e^{\frac{y_i}{2}+\sqrt{-1}\theta_i}$, then

$$\frac{dt_i}{t_i} = \frac{1}{2}dy_i + \sqrt{-1}d\theta_i$$

$$\frac{d\bar{t}_i}{\bar{t}_i} = \frac{1}{2}dy_i + \sqrt{-1}d\theta_i$$

$$\frac{dt_i\wedge d\bar{t}_i}{|t_i|^2} = -\sqrt{-1}dy_i\wedge d\theta_i$$

$$\partial\bar{\partial}u = \sum_{i,j}\frac{\partial^2 u}{\partial y_i\partial y_j}\frac{dt_i\wedge d\bar{t}_j}{t_i\bar{t}_j}$$

$$(\partial\bar{\partial}u)^n = \det(\frac{\partial^2 u}{\partial y_i\partial y_j})dy_1\wedge\ldots\wedge dy_n\wedge d\theta_1\wedge\ldots\wedge d\theta_n$$

Lemma 4.3 *Let $\tilde{F} = e^{\tilde{u}} \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j}$, then $0 < c \leq \tilde{F} \leq C$.*

Proof $e^{-u} \frac{dt_1 \wedge d\bar{t}_1 \wedge ... \wedge dt_n \wedge d\bar{t}_n}{|t_1|^2 ... |t_n|^2} = e^{-\tilde{u}} dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n$ can be extended to a non-vanishing volume form on X. Also

$$\begin{aligned}(\partial\bar{\partial}u)^n &= \det \frac{\partial^2 u}{\partial t_i \bar{\partial} t_j} dt_1 \wedge d\bar{t}_1 \wedge ... \wedge dt_n \wedge d\bar{t}_n \\ &= \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} dy_1 \wedge ... \wedge dy_n \wedge d\theta_1 \wedge ... \wedge d\theta_n\end{aligned}$$

is a non-vanishing volume form, so the quotient of these two volume form must be positive and bounded. which proves the lemma.

Now we can prove the Theorem 1.1.

$$\begin{aligned}\int_X e^{-\alpha\varphi}\omega^n &= \int_X e^{-\alpha\varphi}(\partial\bar{\partial}u)^n \\ &= \int_{R^n} e^{-\alpha\tilde{\varphi}} \det \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} dy_1 ... dy_n \\ &\leq \int_{R^n} e^{-\alpha\tilde{\varphi} - \tilde{u}} dy_1 ... dy_n \\ &\leq \int_{R^n} e^{-\alpha \log \frac{e^{<v,y>}}{\sum e^{<v_i,y>}} - \log(\sum e^{<v_i,y>})} dy_1 ... dy_n \\ &= \int_{R^n} \frac{e^{-\alpha<v,y>}}{(\sum e^{<v_i,y>})^{1-\alpha}} dy_1 ... dy_n\end{aligned}$$

If the stable points $S = \{0\}$, then X is symmetric so that $v = \frac{\sum_{g \in W(X)} g\mu}{m|W(X)|} = 0$ for all $\mu \in L(\Sigma^{(m)})$. Therefore for all $\alpha < 1$ the integral

$$\int_X e^{-\alpha\varphi}\omega^n = \int_{R^n} \frac{1}{(\sum_{i=0}^k e^{<v_i,y>})^{1-\alpha}} dy_1 ... dy_n$$

is finite since every n-dimensional cone $\sigma_j \in \Delta (j = 1, ..., l)$ is generated by a basis of the lattice N and the fact that $N_R = \sigma_1 \cup ... \cup \sigma_l$. This implies $\alpha_G(X) \geq 1$ so that by Tian's theorem[14] X admits Kahler-Einstein metric. This is a result by V. Batyrev and E.N. Selivanova[2].

If $S \neq \{0\}$ then for any $0 \neq v \in S$, we have $w_v \in \partial\Sigma$ related with v satisfying

$$< w_v, v >= -|w_v||v|.$$

The integral

$$\int_{R^n} \frac{e^{-\alpha<v,y>}}{(\sum e^{<v_i,y>})^{1-\alpha}} dy_1...dy_n = \int_{R^n} \left(\frac{e^{-\frac{\alpha}{1-\alpha}<v,y>}}{\sum e^{<v_i,y>}} \right)^{1-\alpha} dy_1...dy_n$$

is finite if

$$-\frac{\alpha}{1-\alpha} v \in int(\Sigma).$$

i.e.

$$< -\frac{\alpha}{1-\alpha} v, w_v >= \frac{\alpha}{1-\alpha} |v||w_v| \leq |w_v|^2$$

Then for all $\alpha < \frac{\min_{0\neq v\in S} \frac{|w_v|}{|v|}}{1+\min_{0\neq v\in S} \frac{|w_v|}{|v|}}$ the integral $\int_X e^{-\alpha\varphi}\omega^n$ is finite. Therefore

$$\alpha_G(X) \geq \frac{\min_{0\neq v\in S} \frac{|w_v|}{|v|}}{1+\min_{0\neq v\in S} \frac{|w_v|}{|v|}}.$$

In order to estimate the upper bound of $\alpha_G(X)$ we will construct a sequence of PSH functions. Suppose $S \neq \{0\}$, then for all α with $1 > \alpha > \frac{\min_{0\neq v\in S} \frac{|w_v|}{|v|}}{1+\min_{0\neq v\in S} \frac{|w_v|}{|v|}}$ we choose $\tilde{\varphi}_\epsilon = \log(\frac{e^{<\tilde{v},y>}+\epsilon}{\sum e^{<v_i,y>}})$ which is increasing and uniformly bounded from above where $\min_{0\neq v\in S} \frac{|w_v|}{|v|}$ is achieved at $\tilde{v} \in S$. Then by Fatou lemma we have

$$\lim_{\epsilon\to 0} \int_X e^{-\alpha\varphi_\epsilon}\omega^n = \int_X e^{-\alpha\varphi_0}\omega^n = \infty$$

which implies $\alpha_G(X) \leq \frac{\min_{0\neq v\in S} \frac{|w_v|}{|v|}}{1+\min_{0\neq v\in S} \frac{|w_v|}{|v|}}$. Combined the above estimates together, we have proved the Theorem 1.1.

Also it's obvious to see that $\min_{0\neq v\in S} \frac{|w_v|}{|v|} \leq 1$ for non-symmetric toric Fano manifold X thus $\alpha_G(X) \leq \frac{1}{2}$. This shows that there doesn't exist any non-symmetric toric Fano manifold such that its α_G-invariant is greater than $\frac{n}{n+1}$ which is a sufficient condition for the existence of Kähler-Einstein metrics.

Now we prove Theorem 1.2 which is a direct corollary of the proof of Theorem 1.1. Define $P_{m,G}(X)$=$\{\varphi \in C^\infty(X,R) | \sup_X \varphi = 0$, φ is G-invariant and there exists a basis $\{S_i^m\}_{1\leq i\leq N_m}$ of $H^0(X, K_X^{-m})$ such that $\omega_g + \partial\overline{\partial}\varphi = \frac{1}{m}\partial\overline{\partial}\log(\sum_{i=0}^{N_m} |S_i^m|^2)\}$, where $N_m + 1 = \dim H^0(X, K_X^{-m})$ and m is large.

We also define for m large, $\alpha_{m,G}(X) = \sup\{\alpha |$ there exists $C > 0$ such that for all $\varphi \in P_{m,G}(X)$, $\int_X e^{-\alpha\varphi} dV \leq \infty\}$.

It's easy to see $\alpha_{m,G}(X)$ is decreasing as m goes to the infinity. By the argument to give the upper bound for the α_G-invariant, we can directly have the following corollary which answers the question proposed by Tian[15] in the special case of toric Fano manifolds.

Corollary 4.3 *If X is a toric Fano manifold, then $\{\alpha_{m,G}(X)\}_m$ is decreasing and stationary. More precisely, $\alpha_{m,G}(X) = \alpha_G(X)$ if $m \geq m_0$, where m_0 is the least positive integer such that $m_0 v \in M$ and v is the minimizer of $\min_{0\neq v\in S} \frac{|w_v|}{|v|}$.*

§5. Multiplier ideal sheaf

In this section we relate the α-invariant on toric Fano manifolds with the method of the multiplier ideal sheaf employed by Nadel[9]. Here we follow the lines in [3].

Theorem 5.5 (Nadel) *Let X be a Fano manifold of dimension n and G be a compact subgroup of the group of complex automorphisms of X. Then X admits a G-invariant Kähler-Einstein metric, unless K_X^{-1} possesses a G-invariant singular hermitian metric $h = h_0 e^{-\varphi}$ (h_0 is a smooth G-invariant metric and φ is a G-invariant function in $L^1_{loc}(X)$), such that the following properties occur.*

1. *h has a semipositive curvature current*
$$\Theta_h = -\frac{i}{2\pi}\partial\overline{\partial}\log h = \Theta_{h_0} + \frac{i}{2\pi}\partial\overline{\partial}\varphi \leq 0.$$
2. *For every $\gamma \in (\frac{n}{n+1}, 1)$, the multiplier ideal sheaf $\mathcal{J}(\gamma\varphi)$ is nontrivial, (i.e. $0 \neq \mathcal{J}(\gamma\varphi) \neq O_X$).*

Theorem 5.6 (Nadel) *Let (X, ω) be a Kähler manifold and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight ϕ with respect to a smooth metric h_0(i.e. $h = h_0 e^{-\phi}$). Assume that the curvature form $\Theta_h(L)$ is positive definite in the sense of currents, i.e. $\Theta_h(L) \geq \epsilon\omega$ for some $\epsilon > 0$. Then we have $H^q(X, K_X \otimes L \otimes \mathcal{J}(\phi)) = 0$ for all $q \geq 1$*

Corollary 5.4 (Nadel) *Let X, G, h and φ be in theorm 4.1. then for all $\gamma \in (\frac{n}{n+1}, 1)$,*

1. *The multiplier ideal sheaf $\mathcal{J}(\gamma\varphi)$ satisfies $H^q(X, \mathcal{J}(\gamma\varphi) = 0$ for all $q \geq 1$.*
2. *The associated subscheme V_γ of structure sheaf $O_{V_\gamma} = O_X/\mathcal{J}(\gamma\varphi)$ is nonempty, distinct from X, G-invariant and satisfies $H^q(V_\gamma, O_{V_\gamma})= C$ for q=0 and vanishes for $q \geq 1$.*

In order to construct Kahler-Einstein metrics it's sufficient to rule out the existence of any G-invariant subscheme with the property (2) in the corollary.
In the case of toric Fano manifolds, we have the following theorem.

Theorem 5.7 *Let X be a toric Fano manifold. If X is not symmetric then there always exists a G-invariant subscheme with the property (2) in the corollary.*

Proof If X is not symmetric then $\alpha_G(X) \leq \frac{1}{2}$ and we can construct G-invariant $\varphi \in L^1_{loc}(X)$ such that for all $\gamma \in (\alpha_G(X), 1)$

$$\int_X e^{-\gamma\varphi}\omega^n = +\infty$$

therefore $J(\gamma\varphi)$ is nontrivial and there exist subschemes V_γ satisfying property (2) of the corollary.

§6. Examples

In this sections we will calculate the α invariants for 2-dimensional toric Fano manifolds.

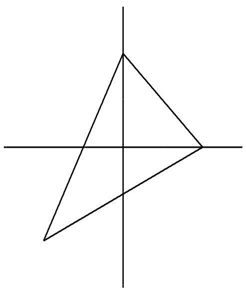
Here (1) (2) (3) (4) are corresponding to CP^2 and CP^2 blow-up at 1, 2 and 3 points and (5) (6) (7) (8) are the corresponding polyhedrons.
CP^2 and CP^2 blow-up at 3 points are symmetric thus the their α_G-invariants are both equal to 1.

For $CP^2\#1\overline{CP^2}$ its stable points of G on the boundary of the polyhedron in (6) is $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ and $\frac{|(1/2,1/2)|}{|(-1/2,-1/2)|} = 1$ then it is easy to see $\alpha_G(CP^2\#1\overline{CP^2}) = \frac{1}{2}$.

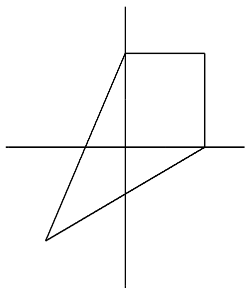
For $CP^2\#2\overline{CP^2}$ its stable points of G on the boundary of the polyhedron in (7) is $(\frac{1}{2}, \frac{1}{2})$ and $(-1, -1)$ and $\frac{|(1/2,1/2)|}{|(-1,-1)|} = \frac{1}{2}$ then it is easy to see $\alpha_G(CP^2\#2\overline{CP^2}) = \frac{1}{3}$.

The above calculation confirms our earlier results[12].

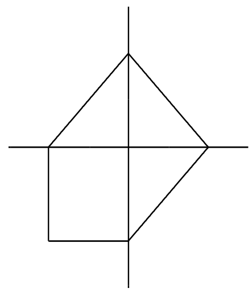
(1)



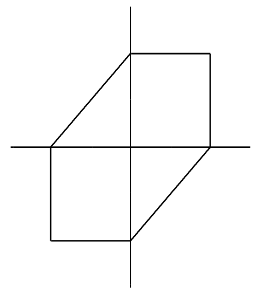
(2)



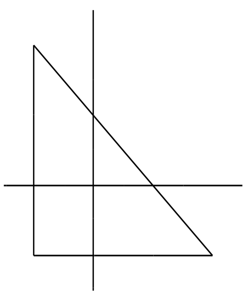
(3)



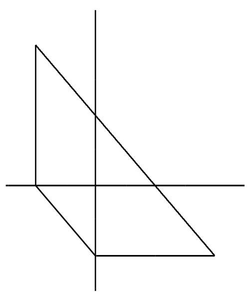
(4)



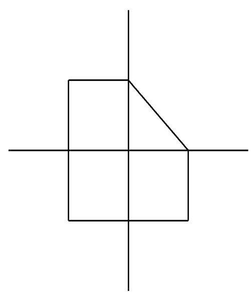
(5)



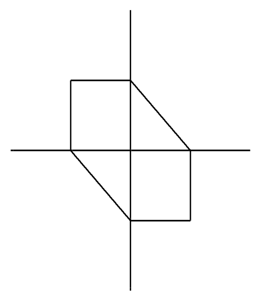
(6)



(7)



(8)



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